An Optimal Life Insurance Policy in the Investment-Consumption Problem in an Incomplete Market[*]

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Abstract

This paper considers an optimal life insurance for a householder subject to mortality risk. The household receives a wage income continuously, which is terminated by unexpected (premature) loss of earning power or (planned and intended) retirement, whichever happens first. In order to hedge the risk of losing income stream by householder's unpredictable event, the household enters a life insurance contract by paying a premium to an insurance company. The household may also invest their wealth into a financial market. The problem is to determine an optimal insurance/investment/consumption strategy in order to maximize the expected total, discounted utility from consumption and terminal wealth. To reflect a real-life situation better, we consider an incomplete market where the householder cannot trade insurance contracts continuously. To our best knowledge, such a model is new in the insurance and finance literature. The case of exponential utilities is considered in detail to derive an explicit solution. We also provide numerical experiments for that particular case to illustrate our results.

1 Introduction

In this paper, we consider an optimal life insurance for a householder subject to mortality risk in a continuous time economy. That is, every trade occurs continuously at time $t \in \mathcal{T}$, $\mathcal{T} := [0, T]$, where T denotes the retirement time of the householder and the current time is 0. The household receives

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1

a wage income at rate $y(t)$ continuously, which is terminated by the householder's unexpected loss (e.g. death) of earning power or intentional retirement, whichever happens first. The mortality risk is formulated as the first occurrence of events in a Poisson process $N = \{N(t); t \geq 0\}$ with intensity process $\lambda = \{\lambda(t); t \geq 0\}$. We denote the random time of that event by τ. In order to hedge the risk of losing the earning power, the household enters a life insurance contract that matures at time T that corresponds to the intended retirement time. In return, the householder pays a premium amount $\theta = \{\theta(t); t \in \mathcal{T}\}$ to an insurance company continuously until time τ or T, whichever happens first. In compensation of the premium, the insurance company pays an insurance amount $\theta(t)(1 + \delta(\tau))$ only on the set $\{\omega : \tau(\omega) \leq T\}$ where $\delta = \{\delta(t); t \in \mathcal{T}\}$ is assumed to be a positive process that is given exogenously. The household can also invest their wealth into a financial market consisting of a riskless security (bank account) and a risky security (stock). The life insurance and investment policies are to be determined by the household. It is noted that the premium amount θ is treated as a part of control variables to obtain an optimal strategy for the household.

Previous treatments of insurance in the literature are either to assume that an insurance amount is given exogenously or to obtain an insurance premium or reserve for insurers or insurance companies. For example, Albizzati and Geman (1994) and Persson and Aase (1997) examined option-like features contained in the given insurance contacts. The fair premium of an equity-linked life insurance contract is calculated in Brennan and Schwartz (1976) and Nielsen and Sandman (1995), while Marceau and Gaillardetz (1999) considered the calculation of the reserves in a stochastic mortality and interest rates environment. See also Iwaki, Kijima and Morimoto (2001) and Iwaki (2002) for determining the insurance premium in a multi-period economy and in a continuous-time economy respectively. In contrast, this paper discusses an optimal insurance from the standpoint of households. To our best knowledge, such a model has not been considered in the insurance literature.

Our problem is formally stated as follows. Let $w(t)$ be the amount invested into the risky security at time t (the process $w = \{w(t); t \in \mathcal{T}\}$ is called a portfolio process), and let $c = \{c(t); t \in \mathcal{T}\}$ be a consumption process, both chosen by the household. Given the initial wealth W_0 of the household, an insurance premium amount process $\theta = \{\theta(t); t \in \mathcal{T}\}$, and an income process $y = \{y(t); t \in \mathcal{T}\}$, the wealth process $W = \{W(t); t \in \mathcal{T}\}$ can be defined so as to maintain the self-financing strategy (to be defined in Section 2) with $W(0) = W_0$. Now, given the discount factor $e^{-\int_0^t \rho(s)ds}$, $t \in \mathcal{T}$, and utility functions $u_i(x)$, $i = 1, 2$, for the consumption and the terminal wealth $W(T)$, respectively, the problem is to find a pair of an optimal insurance premium amount and an investment strategy, $(\hat{\theta}, \hat{w})$, so as to maximize the expected total, discounted utility from consumption and terminal wealth over the budget feasible and admissible consumption, portfolio and insurance.

The problem treated in this paper can be seen as an extension of the security allocation problem originally studied by Merton (1969, 1971). In his model, only a riskless security and a risky security are considered and the problem is to obtain an optimal portfolio selection rule so as to maximize the expected total, discounted utility from consumption through a life cycle. Since then, the model has been extended to various directions to enhance the reality. For example, Bodie, Merton and Samuelson (1992) studied a life time model in which a human capital is considered to represent the present value of the

total wage income obtained in the future. By including the human capital in their security allocation model, they succeeded to explain the relationship between the age of an economic agent and his/her optimal investment strategy. See also He and Pagès (1993), Svensson and Werner (1993), and Karatzas and Shreve (1998) as examples of such extensions.

This paper is organized as follows. In the next section, we formally state our model together with the notation necessary for what follows. Section 3 presents our main results. Namely, the state price density is represented in terms of an intensity process under the equivalent martingale measure, which can be specified uniquely if the market is complete. We solve the problem subject to the constraint that the householder cannot trade insurance contract and prove that an optimal insurance policy along with optimal consumption and investment strategy exists under fairly general conditions. In Section 4, we consider the special case that the household has exponential utility functions for both the consumption and the terminal wealth. If all the data processes are deterministic in time, then a numerical scheme to calculate the optimal solution can be developed. We relegate the detailed proof of our main result to Appendix.

Throughout this paper, all the random variables considered are bounded almost surely (a.s.) to avoid unnecessary technical difficulties. Equalities and inequalities for random variables hold in the sense of a.s.; however, we omit the notation a.s. for the sake of notational simplicity.

2 The Model

Suppose that the current time is 0, and let $T > 0$ be the maturity of an insurance contract. We consider a continuous-time economy in $\mathcal{T} := [0, T]$ that consists of the insurance contract and a financial market. The financial market is assumed to be frictionless and perfectly competitive.[1] Let $Z = \{Z(t); t \in \mathcal{T}\}$ denote the standard Brownian motion on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}_t^Z = \sigma\{Z(s); s \leq t\}, t \in \mathcal{T}$. We denote the $\mathbb{P}$-augmentation of filtration by $\mathbb{F}^Z := \{\mathcal{F}_t^Z; t \in \mathcal{T}\}$. The Brownian motion is the source of randomness other than the time τ.

Let $N = \{N(t); t \in \mathcal{T}\}$ be a Poisson process with intensity process $\lambda = \{\lambda(t); t \in \mathcal{T}\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that $N(0) = 0$ and the intensity process λ is predictable with respect to $\mathbb{F}^Z$. Suppose that the householder is currently alive, and let τ denote the time of householder's premature loss of earning power (e.g. death) defined by the first occurrence of events in the Poisson process N, i.e.

$$\tau = \inf\{t > 0; N(t) = 1\}.$$

Let $\mathcal{F}_t = \mathcal{F}_t^Z \vee \mathcal{F}_t^N, t \in \mathcal{T}$, where $\mathcal{F}_t^N = \sigma\{1_{\{\tau \leq s\}}; s \leq t\}$, and where 1_E denotes the indicator function of event E meaning that $1_E = 1$ if E is true and $1_E = 0$ otherwise. The $\mathbb{P}$-augmentations of filtration are denoted by $\mathbb{F}^N := \{\mathcal{F}_t^N; t \in \mathcal{T}\}$ and $\mathbb{F} := \{\mathcal{F}_t; t \in \mathcal{T}\}$. Clearly, τ is an $\mathbb{F}^N$-stopping time, but not

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[1] A financial market is said to be *frictionless* if the market has no transaction costs, no taxes, and no restrictions on short sales (such as margin requirements), and asset shares are divisible, while it is called *perfectly competitive* if each agent believes that he/she can buy and sell as many assets as desired without changing the market price.

an $\mathbb{F}^Z$-stopping time. It is assumed that $\mathbb{F}$ and $\mathbb{F}^N$ satisfy the usual conditions regarding right-continuity and completeness. The conditional expectation operator given $\mathcal{F}_t$ is denoted by $\mathbb{E}_t$ with $\mathbb{E} = \mathbb{E}_0$.

In this paper, we assume that, given the intensity process λ, the conditional survival probability of τ is given by

$$\mathbb{P}\{\tau > t|\lambda\} = \exp\left\{-\int_0^t \lambda(u)\mathrm{d}u\right\}, \qquad t \in \mathcal{T}. \tag{2.1}$$

That is, the intensity process λ plays the role of the *hazard rate*,

$$\lambda(t) = \lim_{\Delta \downarrow 0} \frac{1}{\Delta}\mathbb{P}\{t < \tau \leq t + \Delta|\tau > t, \mathcal{F}_t^Z\},$$

and such a process is called a Cox process. [2] Note that, in this setting, the infinitesimal increments $\mathrm{d}Z(t)$ and $\mathrm{d}N(t)$ are conditionally independent given $\mathcal{F}_t^Z$. Also, the process $M_\lambda = \{M_\lambda(t), t \in \mathcal{T}\}$ defined by

$$M_\lambda(t) := \int_0^t \mathbf{1}_{\{N(u-)=0\}}[\mathrm{d}N(u) - \lambda(u)\mathrm{d}u] \tag{2.2}$$

is a $\mathbb{F}$-martingale, i.e. the integral $\int_0^t \mathbf{1}_{\{N(u-)=0\}}\lambda(u)\mathrm{d}u$ is the $\mathbb{F}$-compensator (cf. Yashin and Arjas (1988)). To see this, we have

$$M_\lambda(t) = \mathbf{1}_{\{\tau \leq t\}} - \int_0^t \mathbf{1}_{\{N(u-)=0\}}\lambda(u)\mathrm{d}u,$$

since $\int_0^t \mathbf{1}_{\{N(u-)=0\}}\mathrm{d}N(u) = \mathbf{1}_{\{\tau \leq t\}}$. It follows that, for $u > t$,

$$
\begin{aligned}
\mathbb{E}_t[M_\lambda(u)] &= M_\lambda(t) + \mathbb{E}_t\left[\mathbf{1}_{\{t<\tau\leq u\}} - \int_t^u \mathbf{1}_{\{N(s-)=0\}}\lambda(s)\mathrm{d}s\right] \\
&= M_\lambda(t) + \mathbb{E}_t\left[\mathbb{E}_t\left[\mathbf{1}_{\{t<\tau\leq u\}} - \int_t^u \mathbf{1}_{\{\tau>s\}}\lambda(s)\mathrm{d}s\,\middle|\,\mathcal{F}_T^Z\right]\right] \\
&= M_\lambda(t) + \mathbb{E}_t\left[1 - \mathrm{e}^{-\int_t^u \lambda(v)\mathrm{d}v} - \int_t^u \lambda(s)\mathrm{e}^{-\int_t^s \lambda(v)\mathrm{d}v}\mathrm{d}s\right] \\
&= M_\lambda(t),
\end{aligned}
$$

where we have used (2.1) and the fact that

$$\mathbb{E}_t\left[\mathbf{1}_{\{\tau>u\}}\,\middle|\,\mathcal{F}_T^Z\right] = \mathbb{P}_t\{\tau > u|\lambda\}.$$

In the financial market, there is a riskless security whose time t price is denoted by $S_0(t)$. The riskless security evolves according to the differential equation,

$$\frac{\mathrm{d}S_0(t)}{S_0(t)} = r(t)\mathrm{d}t, \qquad t \in \mathcal{T},$$

where $r(t)$ is a positive, predictable process with respect to $\mathbb{F}^Z$. The household can also invest their wealth into a risky security whose time t price is denoted by $S_1(t)$. The risky security evolves according to the stochastic differential equation (abbreviated SDE),

$$\frac{\mathrm{d}S_1(t)}{S_1(t)} = \mu(t)\mathrm{d}t + \sigma(t)\mathrm{d}Z(t), \qquad t \in \mathcal{T}, \tag{2.3}$$

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[2]The process is also known as a doubly stochastic Poisson process; see, e.g., Grandell (1976) for details. Of course, in this case, we have $\mathbb{P}\{\tau > t|\mathcal{F}_t^Z\} = \mathbb{P}\{\tau > t|\mathcal{F}_T^Z\}$ for $t \leq T$.

where $\mu(t)$ and $\sigma(t)$ are progressively measurable processes with respect to $\mathbb{F}^Z$.

As to the life insurance contract, we consider a policy described as follows. Once the householder pays an insurance premium $\theta(t)$ at each time $t \in [0, \tau \wedge T]$ and at time τ if it happen before the contract maturity T, an insurance company makes an insurance payment in the amount of

$$\theta(\tau)(1 + \delta(\tau))$$

at time τ, where $\delta(\tau)$ is a positive multiplier. We assume that $\theta = \{\theta(t); t \in \mathcal{T}\}$ is *the insurance premium amount* and is to be determined by the household and that $\delta = \{\delta(t); t \in \mathcal{T}\}$ is a predictable processes exogenously given in the market.

Let $y = \{y(t); t \in \mathcal{T}\}$ be the *income process*, and let $c = \{c(t); t \in \mathcal{T}\}$ be the *consumption process* chosen by the household. It is assumed that these processes are adapted to $\mathbb{F}$. Let $w(t)$ be the amount invested into the risky security at time t. The process $w = \{w(t); t \in \mathcal{T}\}$ is referred to as a *portfolio process*. Now, given a portfolio process w, consumption process c, insurance premium amount θ, and income process y, the wealth process $W = \{W(t); t \in \mathcal{T}\}$ is defined by

$$
\begin{aligned}
W(t) & = W_0 + \int_0^t \big(r(s)W(s) + y(s)1_{\{N(s-)=0\}} - c(s)\big)\,\mathrm{d}s \\
& \quad + \int_0^t w(s)\left[(\mu(s) - r(s))\mathrm{d}s + \sigma(s)\mathrm{d}Z(s)\right] \\
& \quad + \int_0^t \theta(t)1_{\{N(s-)=0\}}(-r(s)\mathrm{d}s + \delta(s)\mathrm{d}N(s)) - C(t), \qquad t \in \mathcal{T}, \qquad (2.4)
\end{aligned}
$$

where W_0 is a given initial wealth which is assumed to be a positive constant, and $C = \{C(t); t \in \mathcal{T}\}$ is a nonnegative, nondecreasing process which captures the free disposal of wealth. The wealth process W satisfying (2.4) is called *self-financing*, because we have

$$
\begin{aligned}
\mathrm{d}W(t) & = \big(y(t)1_{\{N(t-)=0\}} - c(t)\big)\,\mathrm{d}t + \theta(t)1_{\{N(t-)=0\}}\delta(t)\mathrm{d}N(t) \\
& \quad + w(t)\left[\mu(t)\mathrm{d}t + \sigma(t)\mathrm{d}Z(t)\right] + (W(t) - w(t) - \theta)r(t)\mathrm{d}t - \mathrm{d}C(t),
\end{aligned}
$$

which holds true under the self-financing trading strategy (see, e.g., Duffie (2001)). Rearranging the terms and integrating it over $[0, t]$, we obtain (2.4). Note that $W(T)$ represents the terminal wealth which will be used for their lives after retirement or a bequest to their descendants.

The next definition is similar to the one given by Cuoco (1997).

Definition 1. A consumption and terminal wealth pair $(c, W(T))$ is called *feasible* if $c(t) \geq 0$, $W(t) > -\infty$ for $t \in \mathcal{T}$ and $W(T) \geq 0$.

Recall that the household consumes the wage income and insurance, if any, to maximize the expected total, discounted utility from consumption c and terminal wealth $W(T)$. Let $u_1 : (0, \infty) \to \mathbb{R}$ be the utility function for consumption, and let $u_2 : (0, \infty) \to \mathbb{R}$ be the utility function for the terminal

wealth. It is assumed that these functions are strictly increasing, strictly concave and twice continuously differentiable with properties

$$u_i'(\infty) := \lim_{x \to \infty} u_i'(x) = 0, \quad u_i'(0+) := \lim_{x \downarrow 0} u_i'(x) = \infty, \quad i = 1, 2.$$

Also, in order to represent time-preference of the household, we introduce a *time-discount factor* $e^{-\int_0^t \rho(s)\mathrm{d}s}$, $t \in \mathcal{T}$, where the process $\rho = \{\rho(t), t \in \mathcal{T}\}$ is adapted to $\mathbb{F}$. Let $\mathcal{C}$ be a class of feasible pairs, $(c, W(T))$, that satisfy

$$\mathbb{E}\left[\int_0^T e^{-\int_0^t \rho(s)\mathrm{d}s} u_1^-(c(t))\mathrm{d}t + e^{-\int_0^T \rho(s)\mathrm{d}s} u_2^- (W(T))\right] < \infty, \tag{2.5}$$

where $x^- = \max\{-x, 0\}$. The problem that the household faces is formally stated as follows:

(P) Given the discount process ρ and utility functions $u_i(x)$, $i = 1, 2$, find an optimal consumption/portfolio process $(\hat{c}, \hat{w})$ as well as an optimal insurance premium amount $\hat{\theta}$ to maximize the expected total, discounted utility from consumption and terminal wealth,

$$U(c, W(T)) = \mathbb{E}\left[\int_0^T e^{-\int_0^t \rho(s)\mathrm{d}s} u_1(c(t))\mathrm{d}t + e^{-\int_0^T \rho(s)\mathrm{d}s} u_2(W(T))\right],$$

over the feasible consumption and terminal wealth pairs, $(c, W(T)) \in \mathcal{C}$.

In the next section, to incorporate the real-life constraint that the householder cannot trade insurance contract continuously (see below for details), we shall solve the problem (P) under this constraint (which makes the problem harder.) We shall employ the martingale approach (see, e.g., Karatzas and Shreve (1998) for details).

3 Main Results

To apply the martingale approach, we need to specify the state price density process first: State price density process $\phi = \{\phi(t); t \in \mathcal{T}\}$ is such a process that $\phi(0) = 1$, $0 < \phi(t) < \infty$, and for each $t \in \mathcal{T}$ and for any $s > t$, $s \in \mathcal{T}$,

$$\mathbb{E}_t[\phi(s)S_j(s)] = \phi(t)S_j(t), \qquad j = 0, 1, \tag{3.1}$$

i.e. each process $\{\phi(t)S_j(t), t \in \mathcal{T}\}$, $j = 0, 1$, is a martingale under $\mathbb{P}$. The equivalent martingale measure $\mathbb{Q}$ is then given by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \frac{\phi(T)}{\beta(T)},$$

where

$$\beta(t) = \exp\left\{-\int_0^t r(s)\mathrm{d}s\right\}, \qquad t \geq 0.$$

Here and hereafter, we denote the conditional expectation operator given $\mathcal{F}_t$ under the equivalent martingale measure $\mathbb{Q}$ by $\mathbb{E}_t^{\mathbb{Q}}$ with $\mathbb{E}^{\mathbb{Q}} = \mathbb{E}_0^{\mathbb{Q}}$.

Our first result is simple, but useful for subsequent developments.

Lemma 1.   *1. The state price density is represented as*

$$\phi(t) := \beta(t)\phi^Z(t)\phi^N(t), \quad t \in \mathcal{T},$$

where

$$\phi^N(t) := \left(\frac{\psi(\tau)}{\lambda(\tau)} 1_{\{\tau \le t\}} + 1_{\{\tau > t\}} \right) e^{\int_0^{t \wedge \tau} (\lambda(s) - \psi(s)) \mathrm{d}s} \tag{3.2}$$

with $t \wedge \tau \equiv \min\{t, \tau\}$, and where

$$\phi^Z(t) := \exp\left\{ -\int_0^t \xi(s) \mathrm{d}Z(s) - \frac{1}{2} \int_0^t \xi^2(s) \mathrm{d}s \right\}; \quad \xi(t) := \frac{\mu(t) - r(t)}{\sigma(t)}. \tag{3.3}$$

Here, $\psi = \{\psi(t); t \in \mathcal{T}\}$ is a positive, predictable process with respect to $\mathbb{F}^Z$.

2. The process ψ represents the intensity process under the equivalent martingale measure $\mathbb{Q}$.

Proof. By applying a version of Itô's formula to (3.2), we obtain

$$\mathrm{d}\phi^N(t) = \phi^N(t)1_{\{N(t-)=0\}} \left(\frac{\psi(t)}{\lambda(t)} - 1 \right) [\mathrm{d}N(t) - \lambda(t)\mathrm{d}t].$$

It follows that

$$\frac{\mathrm{d}(\phi(t)/\beta(t))}{\phi(t-)/\beta(t)} = -\xi(t)\mathrm{d}Z(t) + 1_{\{N(t-)=0\}} \left(\frac{\psi(t)}{\lambda(t)} - 1 \right) [\mathrm{d}N(t) - \lambda(t)\mathrm{d}t], \tag{3.4}$$

so that the process $\{\phi(t)/\beta(t); t \in \mathcal{T}\}$ is an exponential martingale under $\mathbb{P}$. The requirement (3.1) can then be verified at once, and thus the first statement is proved. To prove the second statement, define

$$M_\psi(t) := \int_0^t 1_{\{N(u-)=0\}}[\mathrm{d}N(u) - \psi(u)\mathrm{d}u], \qquad t \in \mathcal{T}.$$

Then, by Theorem IIT9 of Brémaud (1981), it suffices to show that

$$M_\psi(t) = \mathbb{E}_t^{\mathbb{Q}}[M_\psi(s)] = \mathbb{E}_t \left[M_\psi(s) \frac{\phi(s)}{\beta(s)} \right], \qquad t < s, s \in \mathcal{T}.$$

But, by the form of ϕ^N and M_ψ, we only have to prove that

$$I := \mathbb{E}_t \left[\left(1_{\{t < \tau \le s\}} - \int_t^{s \wedge \tau} \psi(u)\mathrm{d}u \right) \phi^N(s) \right] = 0.$$

Suppose that $\tau > t$, since otherwise the result is obvious. Now, from (2.1) and (3.2), we obtain

$$
\begin{aligned}
\frac{I}{e^{\int_0^t (\lambda(v) - \psi(v))\mathrm{d}v}} &= \mathbb{E}_t \left[1_{\{t < \tau \le s\}} \frac{\psi(\tau)}{\lambda(\tau)} \left(1 - \int_t^\tau \psi(v)\mathrm{d}v \right) e^{\int_t^\tau (\lambda(v) - \psi(v))\mathrm{d}v} \right] \\
&\quad - \mathbb{E}_t \left[1_{\{\tau > s\}} \left(\int_t^s \psi(v)\mathrm{d}v \right) e^{\int_t^s (\lambda(v) - \psi(v))\mathrm{d}v} \right] \\
&= \int_t^s \psi(u) \left(1 - \int_t^u \psi(v)\mathrm{d}v \right) e^{-\int_t^u \psi(v)\mathrm{d}v}\mathrm{d}u - \left(\int_t^s \psi(v)\mathrm{d}v \right) e^{-\int_t^s \psi(v)\mathrm{d}v},
\end{aligned}
$$

which is equal to zero upon integration by parts. This complete the proof of the lemma. $\qquad\square$

Lemma 1 reveals that the state price density ϕ is determined once the intensity process ψ is specified. The simplest way for this is to assume that the insurance premium is determined so that there exists no arbitrage in the market, and that the insurance premium process $\{P_I(t); t \geq 0\}$ (that in essence determines the amount $\delta(\tau)$) evolves according to the SDE,

$$\mathrm{d}P_I(t) = P_I(t-)1_{\{N(t-)=0\}}\delta(t)\mathrm{d}N(t), \qquad t \geq 0. \tag{3.5}$$

Now, in order for $\phi(t)$ to be the state price density, we must have the process $\{\phi(t)P_I(t); t \in \mathcal{T}\}$ is a martingale under P as in (3.1). To this end, we obtain from (3.4) and (3.5) that, for each $t \geq 0$, if $N(t-) = 0$,

$$\frac{\mathrm{d}(\phi(t)P_I(t))}{\phi(t-)P_I(t-)} = (\delta\psi(t) - r(t))\,\mathrm{d}t - \xi(t)\mathrm{d}Z(t) + \left(\left(\delta(t) + 1\right)\frac{\psi(t)}{\lambda(t)} - 1\right)[\mathrm{d}N(t) - \lambda(t)\mathrm{d}t].$$

So that $\psi(t)$ is given as

$$\psi(t) = \frac{r(t)}{\delta(t)}, \qquad t \geq 0. \tag{3.6}$$

Let $\theta(t)$ be units of insurance which is held by the household at time $t \in \mathcal{T}$. Since it is practically impossible to assume that the household shall trade insurance contracts continuously, in the model, the household is assumed to be restricted to choose, at time 0, $\theta(t)$ to be a constant θ for all $t \in \mathcal{T}$. Therefore, the problem (**P**) is modified as follows: the household select a consumption, portfolio and insurance process $\{(c(t), w(t), \theta(t)), t \in \mathcal{T}\}$ under the constraint that $\theta(t) = \theta, t \in \mathcal{T}$, to maximize the expected total, discounted utility from consumption and terminal wealth. In other words, the household faces the following problem;

(**MP**) Given the discount process ρ and utility functions $u_i(x)$, $i = 1, 2$, find an optimal consumption/portfolio/insurance process $(\hat{c}, \hat{w}, \hat{\theta})$ to maximize the expected total, discounted utility from consumption and terminal wealth,

$$U(c, W(T)) = \mathbb{E}\left[\int_0^T \mathrm{e}^{-\int_0^t \rho(s)\mathrm{d}s} u_1(c(t))\mathrm{d}t + \mathrm{e}^{-\int_0^T \rho(s)\mathrm{d}s} u_2(W(T))\right],$$

over the feasible consumption and terminal wealth pairs, $(c, W(T)) \in \mathcal{C}$ under the budget constraint;

$$\begin{aligned}
W(t) &= W_0 + \int_0^t \left(r(s)W(s) + y(s)1_{\{N(s-)=0\}} - c(s)\right)\mathrm{d}s \\
&\quad + \int_0^t w(s)\left[(\mu(s) - r(s))\mathrm{d}s + \sigma(s)\mathrm{d}Z(s)\right] \\
&\quad + \int_0^t \theta(s)1_{\{N(t-)=0\}}\left(-r(s)\mathrm{d}s + \delta(s)\mathrm{d}N(s)\right) - C(t) \tag{3.7}
\end{aligned}$$

with $\theta(t) = \theta, t \in \mathcal{T}, \theta \in [0, \infty)$.

We solve the problem (**MP**) through 2 steps. First, for a given $\theta \in [0, \infty)$, we solve the problem by applying the auxiliary market approach in the martingale methods for optimal portfolio selection problems (cf.

Chapter 6 of Kartzas and Shreve (1998)). Second, we derive the value of θ which maximizes the value function, which is derived in the first step, of the expected total, discounted utility from consumption and terminal wealth.

To proceed the first step, for a given $\theta \in [0, \infty)$, let $\mathcal{V}$ be a class of predictable stochastic processes defined by

$$\mathcal{V} = \left\{ v(t); \int_0^T |r(t) - v(t)| \mathrm{d}t < \infty, \ t \in \mathcal{T} \right\}.$$

We introduce an auxiliary market where, in place of $\{y(t)1_{\{t<\tau\}}; \ t \in \mathcal{T}\}$, the householder's income process is given as $\{(y(t) - v(t)\theta)1_{\{t<\tau\}}; \ t \in \mathcal{T}\}$ and where the insurance premium process $\{P_I(t); \ t \in \mathcal{T}\}$ evolves according to the SDE;

$$\mathrm{d}P_I(t) = P_I(t-)1_{\{N(t-)=0\}}(v(t)\mathrm{d}t + \delta(t)\mathrm{d}N(t)), \qquad t \geq 0, \tag{3.8}$$

for $v = \{v(t); \ t \in \mathcal{T}\} \in \mathcal{V}$. Note the difference between (3.5) and (3.8), the latter of which has the drift $v(t)P_I(t-)1_{\{N(t-)=0\}}$. By the same arguments as above, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}(\phi(t)P_I(t))}{\phi(t-)P_I(t-)} &= -\xi(t)\mathrm{d}Z(t) + 1_{\{N(t-)=0\}}\left(\delta(t)\psi(t) - r(t) + v(t)\right)\mathrm{d}t \\
&\quad + 1_{\{N(t-)=0\}}\left(\frac{\psi(t)}{\lambda(t)}(\delta(t)+1) - 1\right)[\mathrm{d}N(t) - \lambda(t)\mathrm{d}t],
\end{aligned}
$$

so that the intensity process under the equivalent martingale measure in the auxiliary market, denoted by ψ_v, is given by

$$\psi_v(t) = \frac{r(t) - v(t)}{\delta(t)}, \qquad v \in \mathcal{V}. \tag{3.9}$$

In the following, in order to make the dependence of $v \in \mathcal{V}$ explicit, we denote the state price density by $\phi_v(t) = \beta(t)\phi^Z(t)\phi_v^N(t)$, $t \in \mathcal{T}$, where

$$\phi_v^N(t) = \left(\frac{\psi_v(\tau)}{\lambda(\tau)}1_{\{\tau \leq t\}} + 1_{\{\tau > t\}}\right) \mathrm{e}^{-\int_0^{t \wedge \tau}(\psi_v(s) - \lambda(s))\mathrm{d}s}, \tag{3.10}$$

cf. (3.9). The equivalent martingale measure $\mathbb{Q}_v$ associated with the auxiliary market is given by $\mathrm{d}\mathbb{Q}_v/\mathrm{d}\mathbb{P} = \phi_v(T)/\beta(T)$.

Corresponding to (2.4), we obtain the self-financing wealth process, $W = \{W(t); \ t \in \mathcal{T}\}$, in the

auxiliary market as follows.

$$
\begin{aligned}
W(t) &= W_0 + \int_0^t \big(r(s)W(s) + (y(s) - v(s)\theta)1_{\{N(s-)=0\}} - c(s)\big)\,\mathrm{d}s \\
&\quad + \int_0^t w(s)\left[(\mu(s) - r(s))\mathrm{d}s + \sigma(s)\mathrm{d}Z(s)\right] \\
&\quad + \int_0^t \theta(s)1_{\{N(s-)=0\}}\left[(v(s) - r(s))\mathrm{d}s + \delta(s)\mathrm{d}N(s)\right] - C(t) \\
&= W_0 + \int_0^t \big(r(s)W(s) + y(s)1_{\{N(s-)=0\}} - c(s)\big)\,\mathrm{d}s \\
&\quad + \int_0^t w(s)\left[(\mu(s) - r(s))\mathrm{d}s + \sigma(s)\mathrm{d}Z(s)\right] \\
&\quad + \int_0^t \theta(s)1_{\{N(s-)=0\}}\left[-r(s)\mathrm{d}s + \delta(s)\mathrm{d}N(s)\right] \\
&\quad + \int_0^t v(s)(\theta(s) - \theta)1_{\{N(s-)=0\}}\mathrm{d}s - C(t), \quad t \in \mathcal{T}. \quad (3.11)
\end{aligned}
$$

We note here that $v(t)$ denotes a dual variable with respect to the constraint $\theta(t) = \theta$. That is, we have converted the original constrained problem to an unconstrained problem in the auxiliary market. Given a consumption and terminal wealth pair, $(c, W(T))$, obtained in the auxiliary market, the next result provides a criterion regarding its feasibility in the actual market.

Lemma 2. *A consumption and terminal wealth pair, $(c, W(T))$, is in $\mathcal{C}$ if and only if*

$$
\mathbb{E}^{\mathbb{Q}_v}\left[\int_0^T \beta(t)\big(c(t) - (y(t) - v(t)\theta 1_{\{N(t-)=0\}})\big)\,\mathrm{d}t + \beta(T)W(T)\right] \le W_0 \quad (3.12)
$$

for all $v \in \mathcal{V}$.

Proof. For any $v \in \mathcal{V}$, suppose that $(c, W(T))$ is in $\mathcal{C}$. Then, from (2.4) and Itô's formula, we obtain

$$
\begin{aligned}
\beta(t)W(t) &= W_0 + \int_0^t \beta(s)\big(y(s)1_{\{N(s-)=0\}} - c(s)\big)\,\mathrm{d}s \\
&\quad + \int_0^t \beta(s)w(s)\sigma(s)\mathrm{d}\tilde{Z}(s) \\
&\quad + \int_0^t \beta(s)\theta 1_{\{N(s-)=0\}}(-r(s)\mathrm{d}s + \delta(s)\mathrm{d}N(s)) - \int_0^t \beta(s)\mathrm{d}C(s), \quad (3.13)
\end{aligned}
$$

where $\tilde{Z}(t) := Z(t) + \int_0^t \xi(s)\mathrm{d}s$ is a standard Brownian motion under $\mathbb{Q}_v$ for all $v \in \mathcal{V}$. Now, let $t = T$ and take the expectation on the both sides under the probability measure $\mathbb{Q}_v$. Add and subtract

$$
\int_0^T \beta(s)\theta 1_{\{N(t-)=0\}}\delta(s)\psi_v(s)\mathrm{d}s = \int_0^t \beta(s)\theta 1_{\{N(t-)=0\}}(r(s) - v(s))\mathrm{d}s \quad (3.14)
$$

and use the facts that ψ_v is a $\mathbb{Q}_v$-compensator and C is nondecreasing, we obtain (3.12).

10

To prove the converse, let c be any nonnegative processes, and let η be a nonnegative $\mathcal{F}_T$-measurable random variable. We define $X(t), t \in \mathcal{T}$, by

$$\beta(t)X(t) = \sup_{v \in \mathcal{V}} \mathbb{E}_t^{\mathbb{Q}_v} \left[\int_t^T \beta(s) \left(c(s) - (y(s) - v(s)\theta)1_{\{N(s-)=0\}} \right) \mathrm{d}s + \beta(T)\eta \right].$$

Then, for any $u > t$, $u \in \mathcal{T}$, we have

$$\beta(t)X(t) = \sup_{v \in \mathcal{V}} \mathbb{E}_t^{\mathbb{Q}_v} \left[\int_t^u \beta(s) \left(c(s) - (y(s) - v(s)\theta)1_{\{N(s-)=0\}} \right) \mathrm{d}s + \beta(u)X(u) \right].$$

It follows that the process $\mathcal{M}_v = \{\mathcal{M}_v(t); t \in \mathcal{T}\}$ defined by

$$\mathcal{M}_v(t) := \beta(t)X(t) + \int_0^t \beta(s) \left(c(s) - (y(s) - v(s)\theta)1_{\{N(s-)=0\}} \right) \mathrm{d}s \tag{3.15}$$

is a $\mathbb{Q}_v$-supermartingale for all $v \in \mathcal{V}$. By the Meyer decomposition and the martingale representation theorem (see, e.g., Brémaud (1981) and Karatzas and Shreve (1991)), we conclude that, for each $v \in \mathcal{V}$, there exist a nondecreasing process $A_v = \{A_v(t); t \in \mathcal{T}\}$ with $A_v(0) = 0$, a progressively measurable process $\pi_v^{(1)} = \left\{ \pi_v^{(1)}(t); t \in \mathcal{T} \right\}$ and a predictable process $\pi_v^{(2)} = \left\{ \pi_v^{(2)}(t); t \in \mathcal{T} \right\}$ such that

$$\int_0^T \left| \left(\pi_v^{(1)}(t) \right)^2 + \pi_v^{(2)}(t) \right| \mathrm{d}t < \infty,$$

and satisfy

$$\mathcal{M}_v(t) = X(0) + \int_0^t \pi_v^{(1)}(s)\mathrm{d}\tilde{Z}(s) + \int_0^t \pi_v^{(2)}(s)1_{\{N(s-)=0\}}[\mathrm{d}N(s) - \psi_v(s)\mathrm{d}s] - A_v(t). \tag{3.16}$$

From (3.15), we have

$$\mathcal{M}_v(t) - \int_0^t \beta(s) \left(c(s) - (y(s) - v(s)\theta)1_{\{N(s-)=0\}} \right) \mathrm{d}s$$

$$= \mathcal{M}_0(t) - \int_0^t \beta(s) \left(c(s) - y(s)1_{\{N(s-)=0\}} \right) \mathrm{d}s.$$

This is, $\beta(t)X(t)$ is decomposed to the sum of a supermartingale and a predictable process. Thus, by the uniqueness of both the Meyer decomposition and the canonical decomposition of the special semimartingale (cf., Theorem VI 29.7 and Theorem VI 40.2 of Rogers and Williams (2000)), we can conclude that $\pi_v^{(i)}(s)$, $i = 1, 2$, are independent of v, i.e., $\pi_v^{(i)}(s) = \pi^{(i)}(s)$ say, and

$$\mathrm{d}A_v(t) = \mathrm{d}A_0(t) + v(t) \left(\frac{\pi^{(2)}(t)}{\delta(t)} - \beta(t)\theta 1_{\{N(t-)=0\}} \right) \mathrm{d}t, \quad t \in \mathcal{T}, \tag{3.17}$$

forall $v \in \mathcal{V}$. Here we have used the fact that $\psi_0(t) - \psi_v(t) = \dfrac{v(t)}{\delta(t)}$ from (3.9). Now, from (3.15)–(3.17), we obtain

$$\begin{aligned}
\mathrm{d}X(t) &= \left(r(t)X(t) - c(t) + y(t)1_{\{N(t-)=0\}} \right) \mathrm{d}t \\
&\quad + \frac{\pi^{(1)}(t)}{\beta(t)}\mathrm{d}\tilde{Z}(t) + \frac{\pi^{(2)}(t)}{\beta(t)}1_{\{N(t-)=0\}}[\mathrm{d}N(t) - \psi_0(t)\mathrm{d}t] - \frac{\mathrm{d}A_0(t)}{\beta(t)}. \tag{3.18}
\end{aligned}$$

11

Here, defining

$$w(t) = \frac{\pi^{(1)}(t)}{\beta(t)\sigma(t)}, \tag{3.19}$$

$$\theta(t) = \frac{\pi^{(2)}(t)}{\beta(t)\delta(t)}, \tag{3.20}$$

and $C(t) = \int_0^t \frac{\mathrm{d}A_0(t)}{\beta(t)}$ for $t \in \mathcal{T}$, we obtain

$$
\begin{aligned}
X(t) &= X(0) + \int_0^t \left(r(s)X(s) + y(s)1_{\{N(s-)=0\}} - c(s) \right) \mathrm{d}s \\
&+ \int_0^t w(s)[(\mu(s) - r(s))\mathrm{d}s + \sigma(s)\mathrm{d}Z(s)] \\
&+ \int_0^t \theta(s)1_{\{N(s-)=0\}}[-r(s)\mathrm{d}s + \delta(s)\mathrm{d}N(s)] - C(t), \quad t \in \mathcal{T}.
\end{aligned}
$$

Also, from (3.17) and (3.20), for any $v \in \mathcal{V}$ and any $t \in \mathcal{T}$, we have

$$0 \le \int_0^t v(s)\beta(s)(\theta(t) - \theta)1_{\{N(s-)=0\}}\mathrm{d}s.$$

This leads that $\theta(t) = \theta$ for all $t \in \mathcal{T}$, since otherwise there is some $v \in \mathcal{V}$ such that the right-hand side in the above inequality becomes negative. By a comparison with (2.4), we obtain the desired result with $X(t) = W(t)$, $t \in \mathcal{T}$, and $W(T) = \eta$. $\qquad \square$

According to Lemma 2, the householder's problem becomes

(MP$'$) $\quad \left| \begin{array}{ll} \max & U(c, W(T)), \\ \text{s.t.} & \mathbb{E}^{\mathbb{Q}_v}\left[\int_0^T \beta(t)\left(c(t) - (y(t) - v(t)\theta)1_{\{N(t-)=0\}}\right) \mathrm{d}t + \beta(T)W(T) \right] \le W_0, \quad \forall v \in \mathcal{V}. \end{array} \right.$

We note that, under some regularity conditions on the utility functions, the existence of a solution for the problem (MP$'$) is guaranteed (cf. Theorem 2 of Cuoco (1997)). Now, in order to solve the problem (MP$'$), we consider the following optimization problem:

$$\textbf{(TP)} \quad \max_{\theta \in \mathbb{R}_+} \min_{(\zeta(\theta), v(\theta)) \in \mathbb{R}_{++} \times \mathcal{V}} J(\zeta(\theta), v(\theta)),$$

where

$$
\begin{aligned}
J(\zeta, v) &= \mathbb{E}\left[\int_0^T \tilde{u}_1(\zeta\phi_v(t), t)\mathrm{d}t + \tilde{u}_2(\zeta\phi_v(T), T) \right. \\
&\left. + \zeta\left(W(0) + \int_0^T \phi_v(t)(y(t) - v(t)\theta)1_{\{N(t-)=0\}}\mathrm{d}t \right) \right] \tag{3.21}
\end{aligned}
$$

with

$$\tilde{u}_i(z, t) = \sup_{c \ge 0}\left[\mathrm{e}^{-\int_0^t \rho(s)\mathrm{d}s}u_i(c) - zc \right], \quad t \in \mathcal{T}, \quad i = 1, 2.$$

For each utility function $u_i(x)$, $i = 1, 2$ and each $t \in \mathcal{T}$, we denote by $I_i(x, t)$ the inverse function of $\frac{\mathrm{d}}{\mathrm{d}x}\left[u_i(x)\mathrm{e}^{-\int_0^t \rho(s)\mathrm{d}s}\right]$ with respect to x. Under the assumptions stated above, for each $t \in \mathcal{T}$, the functions $I_i(x, t)$, $i = 1, 2$, exist, and are continuous, strictly decreasing, and map $(0, \infty)$ onto itself with respect to x, with properties $I_i(0+, t) = \infty$ and $I_i(\infty, t) = 0$. Also, it can be readily shown under these conditions that

$$\tilde{u}_i(z, t) = \mathrm{e}^{-\int_0^t \rho(s)\mathrm{d}s}u_i(I_i(z, t)) - zI_i(z, t), \quad t \in \mathcal{T}, \quad i = 1, 2. \tag{3.22}$$

The householder's optimal consumption/wealth process is given by the next theorem. The proof is given in Appendix.

Theorem 1. *For any $c \in (0, \infty)$, suppose that there exist real numbers $a \in (0, \infty)$ and $b \in (0, \infty)$ satisfying $au_i'(c) \geq u_i'(bc)$. Let θ^* be a solution to* (TP) *satisfying*

$$\mathbb{E}^{\mathbb{Q}_{v^*}}\left[\int_0^T \beta(t)I_1(\zeta^*\phi_{v^*}(t), t)\mathrm{d}t + \beta(T)I_2(\zeta^*\phi_{v^*}(T), T)\right] < \infty,$$

where $(\zeta^*, v^*) := \operatorname{argmin}J(\zeta(\theta^*), v(\theta^*))$.

Then, under the conditions stated above, an optimal insurance amount $\hat{\theta}$ is θ^*, and an optimal consumption process $\hat{c}$ and the corresponding wealth process $\hat{W}$ are given, respectively, by

$$\hat{c}(t) = I_1\left(\zeta^*\phi_{v^*}(t), t\right), \quad t \in \mathcal{T}, \tag{3.23}$$

and

$$\hat{W}(t) = \frac{1}{\beta(t)}\mathbb{E}_t^{\mathbb{Q}_{v^*}}\left[\int_t^T \beta(s)\left(\hat{c}(s) - (y(s) - v^*(s)\theta^*)\,1_{\{N(s-)=0\}}\right)\mathrm{d}s + \beta(T)\hat{W}(T)\right], \quad t \in \mathcal{T}, \tag{3.24}$$

with $\hat{W}(T) = I_2(\zeta^*\phi_{v^*}(T), T)$. Furthermore, ζ^* satisfies an equation

$$\mathbb{E}^{\mathbb{Q}_{v^*}}\left[\int_0^T \beta(t)I_1(\zeta^*\phi_{v^*}(t), t)\mathrm{d}t + \beta(T)I_2(\zeta^*\phi_{v^*}(T), T)\right]$$

$$= W_0 + \mathbb{E}^{\mathbb{Q}_{v^*}}\left[\int_0^T \beta(t)(y(t) - v^*(t)\theta^*)1_{\{N(t-)=0\}}\mathrm{d}t\right]. \tag{3.25}$$

An optimal portfolio process $\hat{w}$ is given by (3.19) *with $\hat{w}(t) = w(t)$, $t \in \mathcal{T}$.*

4 The Case of Exponential Utilities

In this section, we consider the case that the household has exponential utility functions for both the consumption and the terminal wealth. Namely, suppose that the utility functions are given by

$$u_i(x) = -\frac{1}{\alpha}e^{-\alpha x}, \quad 0 < x < \infty, \quad i = 1, 2, \tag{4.1}$$

with a constat $\alpha > 0$. It is easily seen from (4.1) that the inverse functions $I_i(x, t)$ are given by

$$I_i(x, t) = -\frac{1}{\alpha}\left(\int_0^t \rho(s)\mathrm{d}s + \ln x\right), \quad i = 1, 2.$$

Observe that $I_i(x, t)$ are continuous, strictly increasing with $I_i(0+, t) = \infty$.

In the following, we also assume for the sake of simplicity that $\mu(t)$, $\sigma(t)$, $r(t)$ and $\rho(t)$ are positive constants,[3] $\mu(t) = \mu$, $\sigma(t) = \sigma$, $r(t) = r$ and $\rho(t) = \rho$ say, while the intensity process λ and the income process y are deterministic functions of time $t \in \mathcal{T}$. We note that, under the assumptions here, both $\xi(t)$ and $\psi(t)$ are such constants that $\xi(t) = \xi$ and $\psi(t) = \psi$, where $\xi = (\mu - r)/\sigma$ and $\psi = r/\delta$. Also, in this case, the Brownian motion Z and the Poisson process N are mutually independent. Recall that the risky security price is driven by the Brownian motion, while the Poisson process determines the householder's death.

In what follows, we derive the optimal insurance amount and portfolio process, $(\hat{\theta}, \hat{w})$, given by Theorem 1 under these assumptions. Calculations to derive the results are simple but tedious. Finally, we give some numerical experiments to demonstrate the usefulness of our results.

4.1 The Optimal Portfolio/Insurance Process

In order to apply the results in Theorem 1, we need to search the value of v^* to solve the problem (TP).

In the exponential utility case, we have from (3.21) and (3.22) that

$$J(\zeta, v) = -\frac{\zeta}{\alpha} \left[\int_0^T e^{-rs}(h(0, s, v) - f(0, s, v))ds + e^{-rT} h(0, T, v) - \alpha W_0 \right], \tag{4.2}$$

where

$$
\begin{aligned}
h(t, s, v) &= 1 - \ln \zeta - \gamma(s - t) - \int_t^s \left(\ln \frac{\psi_v(u)}{\lambda(u)} + \frac{\lambda(u) - \psi_v(u)}{\psi_v(u)} \right) \psi_v(u) e^{-\int_t^u \psi_v(\ell)d\ell} du \\
f(t, s, v) &= \alpha(y(s) - (r - \psi_v(s)\delta)\theta)e^{-\int_t^s \psi_v(u)du},
\end{aligned}
$$

with

$$\gamma = \rho - r + \frac{1}{2}\xi^2.$$

So that, $J(\zeta, v)$ is minimized w.r.t. v if both $h(0, s, v) - f(0, s, v)$ and $h(0, T, v)$ are minimized with respect to $\psi_v(s)$ at each time $s \in \mathcal{T}$. Since, at each time $s \in \mathcal{T}$,

$$\alpha(y(s) - (r - \psi_v(s)\delta)\theta)$$

is an increasing function of $\psi_v(s)$ and

$$\lim_{\psi_v(s) \to 0+} \left(\ln \frac{\psi_v(s)}{\lambda(s)} + \frac{\lambda(s) - \psi_v(s)}{\psi_v(s)} \right) \psi_v(s) = 0, \quad \left(\ln \frac{\psi_v(s)}{\lambda(s)} + \frac{\lambda(s) - \psi_v(s)}{\psi_v(s)} \right) \geq 0, \ \forall \psi_v(s) > 0.$$

both $h(0, s, v) - f(0, s, v)$ and $h(0, T, v)$ are minimized when we set $\psi_v(s) = 0$, $s \in \mathcal{T}$. In other words, an optimal v^* is given by $v^* = \{v(t) = r; t \in \mathcal{T}\}$. Also, from (3.25), the optimal ζ is given by

$$\zeta = \exp\left(-\frac{1}{g_1(0)} \left\{ \gamma g_2(0) + \alpha \int_0^T e^{-rs}(y(s) - r\theta)ds + \alpha W_0 \right\} \right) \tag{4.3}$$

---

[3] An extension to the case that they are deterministic functions of time $t \in \mathcal{T}$ is straightforward.

where

$$g_1(t) = \int_t^T e^{-r(s-t)}ds + e^{-r(T-t)} = \frac{1 - e^{-r(T-t)}}{r} + e^{-r(T-t)},$$

$$g_2(t) = \int_t^T (s-t)e^{-r(s-t)}ds + (T-t)e^{-r(T-t)}$$

$$= \frac{r-1}{r}(T-t)e^{-r(T-t)} + \frac{1 - e^{-r(T-t)}}{r^2}, \quad t \in \mathcal{T}.$$

An optimal insurance amount $\hat{\theta}$ is given by θ which maximizes $J(\zeta(\theta), v^*)$. Since, from (4.2) and (4.3),

$$J(\zeta(\theta), v^*) = -\frac{\zeta(\theta)}{\alpha}g_1(0),$$

and

$$\frac{\zeta(\theta)}{d\theta} = \frac{\alpha}{g_1(0)}r\int_0^T e^{-rs}ds\zeta(\theta),$$

an optimal insurance amount $\hat{\theta}$ is given by $\hat{\theta} = 0$. That is, in this case, it is optimal for the household not to purchase the insurance policy.

From Theorem 1, the optimal consumption process and the optimal terminal wealth are given, respectively, by

$$\hat{c}(t) = -\frac{1}{\alpha}\left(\ln\left(\zeta^*\phi_{v^*}(t)\right) + \rho t\right), \quad t \in \mathcal{T},$$

and

$$\hat{W}(T) = -\frac{1}{\alpha}\left(\ln\left(\zeta^*\phi_{v^*}(T)\right) + \rho T\right),$$

with

$$\zeta^* = \zeta(0) = \exp\left(-\frac{1}{g_1(0)}\left\{\gamma g_2(0) + \alpha\int_0^T e^{-rs}y(s)ds + \alpha W_0\right\}\right).$$

Using (3.24), a tedious algebra then leads to

$$\hat{W}(t) = \hat{c}(t)g_1(t) - \frac{1}{\alpha}\gamma g_2(t) - \int_t^T e^{-r(s-t)}y(s)ds. \tag{4.4}$$

It follows from (4.4) that

$$d\hat{W}(t) = r\hat{W}(t)dt - \left(\hat{c}(t) - y(t)1_{\{N(t-)=0\}}\right)dt + \frac{\xi}{\alpha}g_1(t)d\tilde{Z}(t).$$

Therefore, from (2.4), the optimal portfolio process is given by

$$\hat{w}(t) = \frac{\mu - r}{\sigma^2\alpha}g_1(t), \quad t \in \mathcal{T}.$$

4.2 Numerical example

Our numerical example here is to confirm the analytical result above from the martingale approach by comparing with the dynamic programming approach. When we set constant parameters, the Hamilton-Jacobi-Bellman equation for the value function $V(t, x)$ of the problem

$$\max_{c(s), w(s)} \mathbb{E}_t \left[\int_t^T e^{-\rho(s-t)} u_1(c(s)) ds + e^{-\rho(T-t)} u_2(W(T)) \right]$$

subject to

$$dW(s) = (rW(s) + y1_{\{N(s-)=0\}} - c(s)) ds + w(s)[(\mu - r) ds + \sigma dZ(s)]$$
$$+ \theta 1_{\{N(s-)=0\}}(-r dt + \delta dN(s)) \quad s \in [t, T]$$

is given by

$$V_t - \rho V$$
$$+ \max_{(c,w) \in \mathbb{R}_+ \times \mathbb{R}} \left[\frac{1}{2}(w\sigma)^2 V_{xx} + (rx + y - c + w(\mu - r) - \theta r) V_x + \left(V(t, x + \theta\delta) - V(t, x) \right) \lambda + u_1(c) \right] = 0$$

with the boundary condition $V(T, x) = u_2(x)$. In the case of the exponential utility, we have the explicit solution $V(t, x) = -e^{-A(t)x - B^\theta(t)}$ where

$$A(t) = \left(\frac{1 - e^{-r(T-t)}}{\alpha r} + \frac{e^{-r(T-t)}}{\alpha} \right)^{-1},$$

$$B^\theta(t) = \exp\left(\int_t^T P(s) ds \right) \left\{ \int_t^T -Q^\theta(s) \exp\left(\int_t^s -P(u) du \right) ds + \ln \alpha \right\},$$

with

$$P(t) = -\frac{A(t)}{\alpha},$$

$$Q^\theta(t) = A(t) \left(\frac{1 - \ln A(t)}{\alpha} - y + \theta r \right) - \rho - \frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2 + \lambda(e^{-A(t)\theta\delta} - 1).$$

The parameters are $(\alpha, y, \lambda, \delta, \rho, r, \mu, \sigma, T) = (0.5, 10, 0.01, 100, 0.2, 0.2, 0.3, 0.25, 10)$. We evaluate the value functions at time $t = 0$. The graph (a) shows two value functions $V^0(0, x)$ with $\theta = 0$ and $V^\theta(0, x)$ with $\theta = 1$ and graph (b) shows that V^θ is maximized for all $x \in \mathbb{R}_+$ with $\theta = 0$.

It should be noted that in graph (a) for each $x \in \mathbb{R}_+$, the horizontal difference h of the two graphs that makes

$$V^0(t, x) = V^\theta(t, x - h) \tag{4.5}$$

denotes the buyer's indifference price of the insurance contract whose premium is θ. It is a common technique for pricing in an incomplete market. See, for example, Young (2004) and Egami and Young (2007).
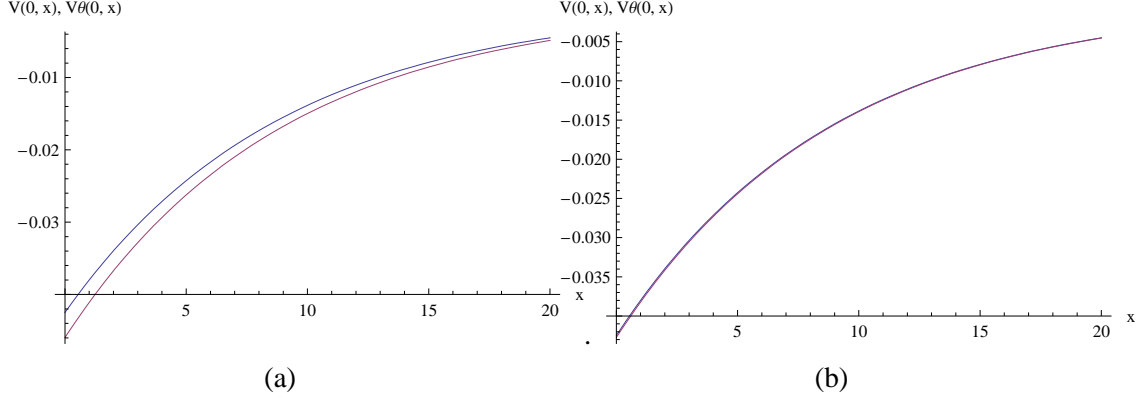
16

Figure 1: (a) The value functions $V^0(0,x)$ with $\theta = 0$ above (blue line) and $V^\theta(0,x)$ with $\theta = 1$ below (red line) are plotted. (b) The value function $V^\theta(0,x)$ is maximized for all $x \in \mathbb{R}_+$ at $\theta = 0$.

5 Concluding Remarks

In this paper, we consider an optimal life insurance for a householder subject to mortality risk in a continuous time economy. In order to hedge the risk to lose the income by an unpredictable event (due to mortality), the household enters a life insurance contract by paying a premium to an insurance company. The household can also invest their wealth into a financial market consisting of a riskless security and a risky security. Under the constraint that the household cannot trade insurance contract, we obtain, using the martingale approach, an optimal insurance/investment strategy so as to maximize the expected total, discounted utility from consumption and terminal wealth. To our best knowledge, such a problem has not been considered in the insurance or finance literature.

The mortality risk is formulated by the first occurrence of events in a Poisson process. We show that the state price density is represented in terms of an intensity process under the equivalent martingale measure, which can be specified uniquely if the market is complete. Also, it is shown that an optimal solution exists under fairly general conditions.

The case of exponential utilities is then examined in detail. Under a set of realistic assumptions, we derive explicit solutions for the optimal investment/insurance policy. In this case, we obtain such a result that it is optimal for the household not to purchase the insurance policy. We conjecture that this result is due to assumptions that the insurance premium, in other words, the state price of the the insurance (P_I in our model: see (3.5) and (3.6)) is determined by the no-arbitrage arguments in the frictionless insurance market and that the mortality risk can not be hedged by purchase of the insurance policy. Hence different specifications on the state price of the insurance may lead to (the household's) different optimal insurance policies. We wish to confirm this conjecture in a more general model (not necessarily the exponential case) that reflects the real insurance market in the near future.

17

A Proof of Theorem 1.

Assume that (ζ^*, v^*) solves (TP), and that

$$\mathbb{E}\left[\int_0^T \phi_{v^*}(t)I_1(\zeta^*\phi_{v^*}(t),t)\mathrm{d}t + \phi_{v^*}(T)I_2(\zeta^*\phi_{v^*}(T),T)\right] < \infty \tag{A.1}$$

holds. In order to prove that $(\hat{c}, \hat{W}(T))$ is optimal, we will proceed in two steps; first we will show that $U(\hat{c}, \hat{W}(T)) \geq U(c, W(T))$ holds for all $(c, W(T)) \in \mathcal{C}$, and then that $(\hat{c}, \hat{W}(T)) \in \mathcal{C}$.

<u>Step 1.</u>   By the assumption there exists $a, b \in (0, \infty)$ such that for each $i = 1, 2$,

$$au_i'(I_i(y,t)) \geq u_i'(bI_i(y,t)) \qquad t \in \mathcal{T}.$$

Applying $I_i(\cdot, t)$ to both sides and iterating, show that for all $a \in (0, \infty)$ there exists a $b \in (0, \infty)$ such that

$$I_i(ay,t) \leq \gamma I_i(y,t), \qquad (y,t) \in (0, \infty) \times \mathcal{T}.$$

Hence, (A.1) implies

$$\mathbb{E}\left[\int_0^T \phi_{v^*}(t)I_1(\zeta_1\phi_{v^*}(t),t)\mathrm{d}t + \phi_{v^*}(T)I_2(\zeta_2\phi_{v^*}(T),T)\right] < \infty \tag{A.2}$$

for all $\zeta_i \in (0, \infty)$, $i = 1, 2$.

By the optimality of ζ^*, we have

$$
\begin{aligned}
0 &= \lim_{\epsilon \to 0} \frac{J(\zeta^* + \epsilon, v^*) - J(\zeta^*, v^*)}{\epsilon} \\
&= \mathbb{E}\left[\int_0^T \lim_{\epsilon \to 0} \frac{\tilde{u}_1((\zeta^* + \epsilon)\phi_{v^*}(t),t) - \tilde{u}_1(\zeta^*\phi_{v^*}(t),t)}{\epsilon}\mathrm{d}t \right. \\
&\qquad + \lim_{\epsilon \to 0} \frac{\tilde{u}_2((\zeta^* + \epsilon)\phi_{v^*}(T),T) - \tilde{u}_2(\zeta^*\phi_{v^*}(T),T)}{\epsilon} \\
&\qquad \left. + W_0 + \int_0^T \phi_{v^*}(t)(y(t) - v^*(t)\theta^* 1_{\{N(t-)=0\}}\mathrm{d}t\right] \\
&= W_0 - \mathbb{E}\left[\int_0^T \phi_{v^*}(t)(\hat{c}(t) - (y(t) - v^*(t)\theta^*)1_{\{N(t-)=0\}})\mathrm{d}t + \phi_{v^*}(T)\hat{W}(T)\right], \quad \text{(A.3)}
\end{aligned}
$$

where the second equality follows Lebesgue's dominated convergence theorem, using (A.2) and the fact that

$$
\begin{aligned}
\left|\frac{\tilde{u}_i((\zeta^* + \epsilon)\phi_{v^*}(t),t) - \tilde{u}_i(\zeta^*\phi_{v^*}(t),t)}{\epsilon}\right| &\leq \frac{\tilde{u}_i((\zeta^* - |\epsilon|)\phi_{v^*}(t),t) - \tilde{u}_i(\zeta^*\phi_{v^*}(t),t)}{|\epsilon|} \\
&\leq \phi_{v^*}(t)I_i((\zeta^* - |\epsilon|)\phi_{v^*}(t),t) \\
&\leq \phi_{v^*}(t)I_i\left(\frac{\zeta^*}{2}\phi_{v^*}(t),t\right),
\end{aligned}
$$

for all $|\epsilon| < \frac{\zeta^*}{2}$ and for each $i = 1, 2$, (because $\tilde{u}_i(\cdot, t)$ is decreasing and convex, $\frac{\partial}{\partial z}\tilde{u}_i(z,t) = -I(z,t)$, and $I(\cdot, t)$ is decreasing). Since by concavity

$$\mathrm{e}^{-\int_0^t \rho(s)\mathrm{d}s}u_i(I_i(z,t),t) - \mathrm{e}^{-\int_0^t \rho(s)\mathrm{d}s}u_i(c,t) \geq z[I_i(z,t) - c] \quad \forall c > 0, z > 0,$$

it then, by evaluating the previous inequality at $z = \zeta^* \phi_{v^*}(t)$ and using the definition of $\hat{c}$ (3.23), follows from (3.12) and (A.3)

$$U(\hat{c}, \hat{W}(T)) - U(c, W(T)) \geq \zeta^* \mathbb{E}\left[\int_0^T \phi_{v^*}(t)(\hat{c}(t) - c(t))\mathrm{d}t + \phi_{v^*}(T)(\hat{W}(T) - W(T))\right] \geq 0.$$

Hence, $(\hat{c}, \hat{W}(T))$ must be optimal provided it is in $\mathcal{C}$.

<u>Step 2.</u>   By the continuity of I_i and ϕ_{v^*}, it is clear that $\int_0^T \hat{c}(t)\mathrm{d}t < \infty$ and that $\hat{W}(T) < \infty$. Also, from the inequality

$$\mathrm{e}^{-\int_0^t \rho(s)\mathrm{d}s} u_i(1) - z \quad \leq \quad \sup_{c \geq 0}\left[\mathrm{e}^{-\int_0^t \rho(s)\mathrm{d}s} u_i(c) - zc\right]$$
$$= \quad \mathrm{e}^{-\int_0^t \rho(s)\mathrm{d}s} u_i(I_i(z,t)) - z I_i(z,t),$$

we have

$$\mathbb{E}\left[\int_0^T \mathrm{e}^{-\int_0^t \rho(s)\mathrm{d}s} u_1^-(\hat{c}(t))\mathrm{d}t + \mathrm{e}^{-\int_0^T \rho(s)\mathrm{d}s} u_2^-(\hat{W}(T))\right]$$
$$\leq \quad \int_0^T \mathrm{e}^{-\int_0^t \rho(s)\mathrm{d}s} u_1^-(1)\mathrm{d}t + \mathrm{e}^{-\int_0^T \rho(s)\mathrm{d}s} u_2^-(1) + \zeta^* \mathbb{E}\left[\int_0^T \phi_{v^*}(t)\mathrm{d}t + \phi_{v^*}(T)\right] < \infty,$$

which shows that $(\hat{c}, \hat{W}(T))$ satisfies the requirement (2.5). We are only left to show that there exists an a admissible portfolio/insurance process $(\hat{w}, \theta^*)$ financing $(\hat{c}, \hat{W}(T))$. Define a process $X = \{X(t); t \in \mathcal{T}\}$ by

$$X(t) \quad := \quad \phi_{v^*}(t)^{-1} \mathbb{E}_t\left[\int_t^T \phi_{v^*}(s)(\hat{c}(s) - (y(s) - v^*(s)\theta^*)1_{\{N(s-)=0\}})\mathrm{d}s + \phi_{v^*}(T)\hat{W}(T)\right]$$
$$= \quad \beta(t)^{-1} \mathbb{E}_t^{Q_{v^*}}\left[\int_t^T \beta(s)(\hat{c}(s) - (y(s) - v^*(s)\theta^*)1_{\{N(s-)=0\}})\mathrm{d}s + \beta(T)\hat{W}(T)\right].$$

Clearly, $X(T) = \hat{W}(T)$, and X is bounded below (because of boundedness of y and θ^* by the assumption). Also, it follows from the martingale representation theorem that there exists a process (π_1, π_2) with $\int_0^T |\pi_1(t)^2 + \pi_2(t)| \mathrm{d}t < \infty$ such that

$$\beta(t)X(t) + \int_0^t \beta(s)(\hat{c}(s) - (y(s) - v^*(s)\theta^*)1_{\{N(s-)=0\}})\mathrm{d}s$$
$$= \quad X(0) + \int_0^t \pi_1(s)\mathrm{d}\tilde{Z}(s) + \int_0^t \pi_2(s)1_{\{N(t-)=0\}}[\mathrm{d}N(s) - \psi_{v^*}(s)\mathrm{d}s]. \qquad (A.4)$$

Define the portfolio/insurance process (w, θ) by

$$\begin{cases} w(t) &= \frac{\pi_1(t)}{\beta(t)\sigma(t)}, \\ \delta(t)\theta(t) &= \frac{\pi_2(t)}{\beta(t)}. \end{cases} \qquad (A.5)$$

19

Using (A.4) and Itô's lemma shows that

$$
\begin{aligned}
X(t) &= X(0) + \int_0^t (r(s)X(s) + (y(s) - v^*(s)\theta^*)1_{\{N(s-)=0\}} - \hat{c}(s))\mathrm{d}s \\
&\quad + \int_0^t \beta(s)^{-1}\pi_1(s)\mathrm{d}\tilde{Z}(s) + \int_0^t \beta(s)^{-1}\pi_2(s)[\mathrm{d}N(s) - \psi_{v^*}(s)\mathrm{d}s] \\
&= X(0) + \int_0^t (r(s)X(s) + (y(s) - v^*(s)\theta^*)1_{\{N(s-)=0\}} - \hat{c}(s))\mathrm{d}s \\
&\quad + \int_0^t w(s)[(\mu(s) - r(s))\mathrm{d}s + \sigma(s)\mathrm{d}Z(s)] + \int_0^t \delta(s)\theta(s)1_{\{N(s-)=0\}}[\mathrm{d}N(s) - \psi_{v^*}(s)\mathrm{d}s] \\
&= X(0) + \int_0^t (r(s)X(s) + y(s)1_{\{N(s-)=0\}} - \hat{c}(s))\mathrm{d}s \\
&\quad + \int_0^t w(s)[(\mu(s) - r(s))\mathrm{d}s + \sigma(s)\mathrm{d}Z(s)] + \int_0^t \theta(s)1_{\{N(s-)=0\}}[-r(s)\mathrm{d}s + \delta(s)\mathrm{d}N(s)] \\
&\quad + \int_0^t [\theta(s) - \theta]v^*(s)1_{\{N(s-)=0\}}\mathrm{d}s.
\end{aligned}
$$

A comparison with (2.4) then reveals that only if we verify that $\theta(t) = \theta^*$ for all $t \in \mathcal{T}$, the proof that $(\hat{c}, \hat{W}(T)) \in \mathcal{C}$ with $\hat{W}(t) = X(t)$, $t \in \mathcal{T}$, is completed.

Fix arbitrary $v \in \mathcal{V}$ and define stochastic processes κ and $\bar{\kappa}$ by, respectively,

$$
\kappa(t) := \int_0^t \frac{1}{\delta(s)} \frac{v(s) - v^*(s)}{\psi_{v^*}(s)} 1_{\{N(s-)=0\}}(\mathrm{d}N(s) - \psi_{v^*}(s)\mathrm{d}s), \tag{A.6}
$$

and

$$
\bar{\kappa}(t) := \int_0^t \frac{v(s) - v^*(s)}{\delta(s)} 1_{\{N(s-)=0\}}\mathrm{d}s, \quad t \in \mathcal{T},
$$

as well as the sequence of stopping times

$$
\tau_n := T \wedge \inf\left\{ t \in \mathcal{T} : |\kappa(t)| + |\bar{\kappa}(t)| + \left| \frac{v(t) - v^*(t)}{r(t) - v^*(t)} \right| \ge n \right\}, \quad n = 1, 2, \cdots.
$$

Then $\tau_n \uparrow T$. Also, letting

$$
v_{\epsilon,n}(t) := v^*(t) + \epsilon[v(t) - v^*(t)]1_{\{t \le \tau_n\}}
$$

for $\epsilon \in \left(0, \frac{1}{nm}\right)$ with some $m > 1$. Clearly $v_{\epsilon,n} = \{v_{\epsilon,n}(t); t \in \mathcal{T}\}$ belongs to $\mathcal{V}$, and therefore

$$
0 \le \frac{1}{\epsilon}[J(\zeta^*, v_{\epsilon,n}) - J(\zeta^*, v^*)] = \mathbb{E}[Y_n^\epsilon],
$$

20

where

$$\epsilon\zeta^* \cdot Y_n^\epsilon := \int_0^T [\tilde{u}_1(\zeta^*\phi_{v_{\epsilon,n}}(t), t) - \tilde{u}_1(\zeta^*\phi_{v^*}(t), t)]dt + \tilde{u}_2(\zeta^*\phi_{v_{\epsilon,n}}(T), T) - \tilde{u}_2(\zeta^*\phi_{v^*}(T), T)$$

$$+ \zeta^* \int_0^T [\phi_{v_{\epsilon,n}}(t) - \phi_{v^*}(t)]y(t)1_{\{N(t-)=0\}}dt - \zeta^* \int_0^T [v_{\epsilon,n}(t)\phi_{v_{\epsilon,n}}(t) - v^*(t)\phi_{v^*}(t)]\theta^*1_{\{N(t-)=0\}}dt$$

$$= \int_0^T [\tilde{u}_1(\zeta^*\phi_{v_{\epsilon,n}}(t), t) - \tilde{u}_1(\zeta^*\phi_{v^*}(t), t)]dt + \tilde{u}_2(\zeta^*\phi_{v_{\epsilon,n}}(T), T) - \tilde{u}_2(\zeta^*\phi_{v^*}(T), T)$$

$$+ \zeta^* \int_0^T [\phi_{v_{\epsilon,n}}(t) - \phi_{v^*}(t)](y(t) - v^*(t)\theta^*)1_{\{N(t-)=0\}}dt$$

$$+ \zeta^*\epsilon \int_0^{\tau_n} (v^*(t) - v(t))\phi_{v_{\epsilon,n}}(t)\theta^*1_{\{N(t-)=0\}}dt.$$

Introduce the ratio

$$R^\epsilon(t) \quad := \quad \frac{\phi_{v_{\epsilon,n}}(t)}{\phi_{v^*}(t)}$$

$$= \quad \exp\left\{\int_0^t \ln\left(\frac{\psi_{v_{\epsilon,n}}(u)}{\psi_{v^*}(u)}\right)1_{\{N(u-)=0\}}dN(u) - \int_0^t (\psi_{v_{\epsilon,n}}(t) - \psi_{v^*}(u))1_{\{N(u-)=0\}}du\right\}$$

$$= \quad e^{\epsilon\bar{\kappa}(t\wedge\tau_n)}(1 - \epsilon(\kappa(t \wedge \tau_n) + \bar{\kappa}(t \wedge \tau_n)))$$

by using (3.9) with $v = v_{\epsilon,n}$ and (3.10). We have then the bounds for R^ϵ:

$$0 < e^{-\frac{1}{m}}\left(1 - \frac{1}{m}\right) \le e^{-\epsilon n}(1 - \epsilon n) \le R^\epsilon(t) \le e^{\epsilon n}(1 + \epsilon n) \le e^{\frac{1}{m}}\left(1 + \frac{1}{m}\right),$$

as well as the upper bounds for the random variable Y_n^ϵ:

$$Y_n^\epsilon \quad \le \quad Q_n := \int_0^T \phi_{v^*}(t)\frac{1 - R^\epsilon(t)}{\epsilon}I_1(\zeta^* e^{-\text{sgn}(t)\epsilon n}(1 - \text{sgn}(t)\epsilon n)\phi_{v^*}(t), t)dt$$

$$+ \phi_{v^*}(T)\frac{1 - R^\epsilon(T)}{\epsilon}I_2(\zeta^* e^{-\text{sgn}(T)\epsilon n}(1 - \text{sgn}(T)\epsilon n)\phi_{v^*}(T), T)$$

$$- \int_0^T \phi_{v^*}(t)\frac{1 - R^\epsilon(t)}{\epsilon}(y(t) - v^*(t)\theta^*)1_{\{N(t-)=0\}}dt$$

$$+ \int_0^{\tau_n} (v^*(t) - v(t))\phi_{v^*}(t)R^\epsilon(t)\theta^*1_{\{N(t-)=0\}}dt,$$

$$Y_n^\epsilon \quad \le \quad Y_n := \sup_{\epsilon\in(0,1/mn)}\frac{1 - e^{-\epsilon n}(1 - \epsilon n)}{\epsilon}\left[\int_0^T \phi_{v^*}(t)I_1\left(\zeta^* e^{-\frac{1}{m}}\left(1 - \frac{1}{m}\right)\phi_{v^*}(t), t\right)dt\right.$$

$$+ \phi_{v^*}(T)I_2\left(\zeta^* e^{-\frac{1}{m}}\left(1 - \frac{1}{m}\right)\phi_{v^*}(T), T\right) + \int_0^T \phi_{v^*}(t)|y(t) - v^*(t)\theta^*|1_{\{N(t-)=0\}}dt\bigg]$$

$$+ e^{\frac{1}{m}}\left(1 + \frac{1}{m}\right)\int_0^{\tau_n} |v^*(t) - v(t)|\phi_{v^*}(t)\theta^*1_{\{N(t-)=0\}}dt,$$

where

$$\text{sign}(t) = \begin{cases} 1; & 1 - R^\epsilon(t) \ge 0, \\ -1; & 1 - R^\epsilon(t) < 0, \end{cases}$$

We have used the mean-value theorem applying to $I_i(y, t) = -\frac{\partial}{\partial y}\tilde{u}_i(y, t)$, $i = 1, 2$ with end points $\phi_{\epsilon,n}(t)$ and $\phi_{v^*}(t)$ and the fact $I_i(y, t)$ is decreasing in y for all $t \in \mathcal{T}$.

Since the random variable Y_n is integrable, then by Fatou's lemma, we obtain

$$
\begin{aligned}
0 \;\le\;& \overline{\lim}_{\epsilon\downarrow 0}\mathbb{E}[Y_n^\epsilon] \le \mathbb{E}[\overline{\lim}_{\epsilon\downarrow 0}Y_n^\epsilon] \le \mathbb{E}[\overline{\lim}_{\epsilon\downarrow 0}Q_n^\epsilon]\\
=\;& \mathbb{E}\Bigg[\int_0^T \phi_{v^*}(t)\kappa(t\wedge\tau_n)(\hat c(t)-(y(t)-v^*(t)\theta^*)1_{\{N(t-)=0\}})\mathrm dt + \phi_{v^*}(T)\kappa(\tau_n)\hat W(T)\\
&+\int_0^{\tau_n}(v^*(t)-v(t))\phi_{v^*}(t)\theta^*1_{\{N(t-)=0\}}\mathrm dt\Bigg]\\
=\;& \mathbb{E}\Bigg[\int_0^{\tau_n}\phi_{v^*}(t)\kappa(t)(\hat c(t)-(y(t)-v^*(t)\theta^*)1_{\{N(t-)=0\}})\mathrm dt\\
&+\kappa(\tau_n)\mathbb{E}_{\tau_n}\Bigg[\int_{\tau_n}^T\phi_{v^*}(t)(\hat c(t)-(y(t)-v^*(t)\theta^*)1_{\{N(t-)=0\}})\mathrm dt + \phi_{v^*}(T)\hat W(T)\Bigg]\\
&+\int_0^{\tau_n}\phi_{v^*}(t)(v^*(t)-v(t))\theta^*1_{\{N(t-)=0\}}\mathrm dt\Bigg]\\
=\;& \mathbb{E}\Bigg[\int_0^{\tau_n}\phi_{v^*}(t)\kappa(t)(\hat c(t)-(y(t)-v^*(t)\theta^*)1_{\{N(t-)=0\}})\mathrm dt + \kappa(\tau_n)\phi_{v^*}(\tau_n)\hat W(\tau_n)\\
&+\int_0^{\tau_n}\phi_{v^*}(t)(v^*(t)-v(t))\theta^*1_{\{N(t-)=0\}}\mathrm dt\Bigg]. \tag{A.7}
\end{aligned}
$$

We used the definition of $\hat W(t)$ (3.24) in the third equality. On the other hand, using (A.4)-(A.6) and Itô's lemma shows that

$$
\begin{aligned}
&\beta(\tau_n)\kappa(\tau_n)W(\tau_n)+\int_0^{\tau_n}\beta(s)\kappa(s)(\hat c(s)-(y(s)-v^*(s)\theta)1_{\{N(s-)=0\}})\mathrm ds\\
=\;&\int_0^{\tau_n}\beta(s)W(s)\frac{v(s)-v^*(s)}{\delta(s)\psi_{v^*(s)}}1_{\{N(s-)=0\}}[\mathrm dN(s)-\psi_{v^*}(s)\mathrm ds]\\
&+\int_0^{\tau_n}\beta(s)\theta(s)\frac{v(s)-v^*(s)}{\psi_{v^*(s)}}1_{\{N(s-)=0\}}[\mathrm dN(s)-\psi_{v^*}(s)\mathrm ds]\\
&+\int_0^{\tau_n}\beta(s)\kappa(s)\delta(s)\theta(s)[\mathrm dN(s)-\psi_{v^*}(s)\mathrm ds]+\int_0^{\tau_n}\beta(s)\kappa(s)w(s)\sigma(s)\mathrm d\tilde Z(s)\\
&+\int_0^{\tau_n}\beta(s)\theta(s)(v(s)-v^*(s))1_{\{N(s-)=0\}}\mathrm ds.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
&\mathbb{E}\Bigg[\int_0^{\tau_n}\phi_{v^*}(s)\kappa(s)(\hat c(s)-(y(s)-v^*(s)\theta^*)1_{\{N(s-)=0\}})\mathrm ds + \phi_{v^*}(\tau_n)\kappa(\tau_n)W(\tau_n)\Bigg]\\
=\;&\mathbb{E}\Bigg[\int_0^{\tau_n}\phi_{v^*}(s)\theta(s)(v(s)-v^*(s))1_{\{N(s-)=0\}}\mathrm ds\Bigg]. \tag{A.8}
\end{aligned}
$$

Substituting (A.8) into (A.7) gives

$$
0 \;\le\; \lim_{\epsilon\downarrow 0}\frac{J(\zeta^*,v_{\epsilon,n})-J(\zeta^*,v^*)}{\zeta^*\epsilon}\le \mathbb{E}\Bigg[\int_0^{\tau_n}\phi_{v^*}(s)(\theta(s)-\theta^*)(v(s)-v^*(s))1_{\{N(s-)=0\}}\mathrm ds\Bigg].
$$

Taking v as $v=v^*+\rho$ with $\rho\in\mathcal V$, it follows that, for arbitrary $v\in\mathcal V$,

$$
\mathbb{E}\Bigg[\int_0^{\tau_n}\phi_{v^*}(s)(\theta(s)-\theta^*)\rho(s)1_{\{N(s-)=0\}}\mathrm ds\Bigg]\ge 0. \tag{A.9}
$$

(A.9) leads to the stronger statement

$$(\theta(t) - \theta^*)\rho(t) \geq 0, \quad t \in \mathcal{T}. \tag{A.10}$$

Indeed, suppose that, for some $t \in \mathcal{T}$, the set $\mathcal{A} = \{\omega \in \Omega; (\theta(t) - \theta^*)\rho(t) < 0\}$ had positive probability for some $\rho \in \mathcal{V}$ such that $v^* + \rho \in \mathcal{V}$. Then by selecting $\rho' = \rho 1_{\mathcal{A}}$, we have $\rho' \in \mathcal{V}$, $v^* + \rho' \in \mathcal{V}$, and

$$\mathbb{E}\left[\int_0^{\tau_n} \phi_{v^*}(s)(\theta(s) - \theta^*)\rho'(s)1_{\{N(s-)=0\}}\,\mathrm{d}s\right] < 0,$$

contradicting (A.9). From (A.10), we obtain

$$(\theta(t) - \theta^*)\rho(t) \geq - \sup_{\theta(t)=\theta^*} \{(\theta(t) - \theta^*)\rho(t)\} \equiv 0, \quad \forall\, \rho \in \mathcal{V}.$$

Therefore, from Theorem 13.1 of Rockafellar (1970), we can conclude that $\theta(t) = \theta^*$, $t \in \mathcal{T}$. $\qquad\square$

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